Convergence to Equilibrium

The object for today is to prove

Theorem

For $(X_t)_{t\geq 0}$ *an irreducible positive recurrent Markov chain, for each* $i, j \in I$,

$$P(X(t) = j | X_0 = i) \to \pi(j)$$

as t *tends to infinity where* π *is the unique invariant distribution.*

Observe: unlike the discrete case there is no need for any aperiodicity assumption.

Useful Discrete Processes

Recall that given $h > 0, Z^h = Z$ is the discrete time markov chain

$$Z_n^h = X_{nh}$$

We will suppress the superscript h if we are only dealing with a single fixed h.

It was shown that X is irreducible if and only if Z is irreducible. Notice that whatever value h has $P_{ii}^Z > e^{-q_i h}$ so Z is aperiodic. In fact we can also see this by the fact that $\forall i, j \ (P^Z)_{ij}^n = P(X_{nh} = j | X_0 = i) > 0$. We also have that π, the invariant distribution for the continuous time Markov chain is the unique invariant distribution forP_h (or equivalently $Z_.$).

So from discrete time theory

$$P_{ij}(nh) \ \to \lambda^h(j)$$

for λ^h stationary for transition probability $P(h)$.

End of Proof

Let us write $h^r = h2^{-r}$ (so $h = h^0$). Again for every h_r

$$P_{ij}(nh^r) \to \lambda^h(j)$$

as n tends to infinity. We use the elementary inequalities for $nh^r \le t \le (n+1)h^r$

$$e^{q_j h^r} P_{ij}((n+1)h^r) \;\geq\; P_{ij}(t) \geq e^{-q_j h^r} P_{ij}(nh^r)$$

This shows that

$$e^{-q_j h^r}\pi(j) \leq \liminf_{t\to\infty} P_{ij}(t) \leq \limsup_{t\to\infty} P_{ij}(t) \leq e^{q_j h^r}\pi(j)$$

But r is arbitrarily large!

RENEWAL PROCESSES

The theory begins with an example. The lifetimes of different light bulbs are i.i.d. positive random variables $X_1, X_2, \cdots$. When a bulb dies it is immediately replaced by another (in dependent) lightbulb. Let $N(t)$be the number of lightbulbs replaced by time t. Write $S_r = \sum_1^r X_k$ (so $S_{N(t)}$ is the time of the last replacement at time t)
Some questions. For t large what can we say about

- $t - S_{N(t)}$, the current lifetime of the bulb in use $= L(t)$
- $S_{N(t)+1} - t$, the remaining lifetime of the bulb in use,$= R(t)$
- $S_{N(t)+1} - S_{N(t)}$ the full lifetime of the current lightbulb $= K(t)$

.

Remark: If the X_i take integer values ,resp. rational values, then $K(t)$ must also take these values. This is not the case with $L(t)$ or $R(t)$. Obviously if X_1 only takes integer values, no limit is posiible for $L(t)$ or $R(t)$ but we will see that generally the limits do exist. .

Two integer examples

First consider $K(t)$ for X_i which take two values with equal probability 1 and $1,000,000$. There is an integer structure for $K(t)$ We interpret $K(n)$ to equal $K(n+\frac{1}{2})+\frac{1}{2}$ for positive integer n. We do not know that $P(K(t)=1)$ has a limit as t becomes large, or that $\frac{1}{t}\int_0^t I_{K(s)=1} ds$ converges to this limit. But suppose we know this. We expect that for large m m about half the $X_i i \leq m$ will be 1 and about half will be $1,000,000$. So for t large, we will have had (to first order) about $t/(2E[X_1])$ bulbs of lifetime 1 and about this number with duration $1,000,000$. So (to first order) the proportion of time s with $K(s)=1$ should be around $\frac{1}{1{,}000{,}001}$. So if the above limits exist then $\lim_t P(K(t)=1)=\frac{1}{1{,}000{,}001}$. Which is very different from $\frac{1}{2}$.

Second integer example

. Recall the following example. We have a law on the positive integers $I = \{1, 2, 3, \cdots\}$, ν which defines the following Markov chain transition probabilities on I: for $n > 1$, $p(n, n-1) = 1$, $p(1, n) = \nu(n)$. For a good fit with lightbulbs, I will take $X(0)$ to have law ν. Then we can see the time after X has value 1 as lightbulb change times. $X(n)$ is simply $R(n)$ for the corresponding lightbulb process. But we know that if

$$\mu = \sum_n n\nu(n) < \infty$$

then our chain $X_.$ is positive recurrent with invariant distribution

$$\pi(n) = \frac{\sum_{m \geq n} \nu(m)}{\mu} = \lim_k P(R(k) = n)$$

and also

$$\lim_{k \to \infty} P(R(k) > n) = \frac{1}{\mu} \sum_{m \geq n} (1 - F(m))$$

Law of large Numbers

The X_i are positive random variables. So the expectation $\mu = E[X_i]$ is well defined (possibly it is infinite). Thus $\frac{S_n}{n}$ converges a.s. to μ. So $\frac{S_{n+1}}{n} = \frac{S_{n+1}}{n+1}\frac{n+1}{n}$ converges to 1 as n tends to infinity. Thus as t becomes large

$$\frac{S_{N(t)+1}}{N(t)} \text{ and } \frac{S_{N(t)}}{N(t)}$$

both converge to μ. But $S_{N(t)} \leq t \leq S_{N(t)+1}$, so $\frac{t}{N(t)}$ does too. Hence

$$\frac{N(t)}{t} \to \frac{1}{\mu}.$$

Let $m(t) = E[N(t)]$. Since $\frac{N(t)}{t} \to \frac{1}{\mu}$, we have $\frac{m(t)}{t} \to \frac{1}{\mu}$.

The Renewal Equation

Let $F(= F_X)$ be the distribution function of the $X_i : F(x) = P(X_1 \leq x)$. We begin with finding a family of equations satisfied by quantities $A(t)$ associated with our random variables X_i. If we wish to understand the limiting distribution of, say, $K(t)$ this reduces to understanding for every positive x the values $P(K(t) > x)$. We fix x and put $A(t) = P(K(t) > x)$.

Now the renewal equation reduces to

$$P(K(t) > x) = E\left[P(K(t) > x | X_1)\right]$$

There are two cases

- $X_1 = y \le t$: $P(K(t) > x | X_1) = A(t - y)$
- $X_1 = y > t$: $P(K(t) > x | X_1) = I_{X_1 > x}$

.

Taking the expectation (over X_1) gives

$$A(t) = \int_0^t A(t - y) dF(y) + (1 - F(x \vee t))$$

A Second Example

We can repeat this argument for $R(t)$. Again we fix x and we look at

$$A(t) = P(R(t) > x)$$

Again we use that the probability is the expectation of the conditional probability: $P(R(t) > x) = E\left[P(R(t) > x|X_1)\right]$. This time

- $X_1 = y \leq t$: $P(R(t) > x|X_1) = A(t-y)$
- $X_1 = y > t$: $P(R(t) > x|X_1) = I_{X_1 > t+x}$

. and so

$$A(t) = \int_0^t A(t-y)dF(y) + (1 - F(t+x))$$

In both (and in other cases) we wish to investigate a quantity $A(t)$ satisfying a RENEWAL EQUATION: for some probability law $dF(x)$ and some function $h(x)$

$$A(t) = \int_0^t A(t-y)dF(y) + h(t)$$

Return to $m(t)$

We wish to provide a convergence result for $m(t) = E[N(t)]$. What might this be? $m(.)$ is a function tending to infinity as t tends to infinity. One thing to look at is (for $h > 0$ fixed) $A(t) = m(t) - m(t-h)$ Two observations

$$A(t) = \int_0^t A(t-y)dF(y) + F(t) - F(t-h)$$

Secondly we can write $N(t)$ as $\sum_n I_{S_n \leq t}$. Taking expectations we obtain

$$m(t) = \sum_n P(S_n \leq t) = \sum_n F_n(t)$$

where F_n is the distribution function of S_n.